Problems 1 and 2 below use a problem from HW 6 #9 about **automorphisms** of groups, restated below. You may want to reference the Fact in your proofs.

Definition 1: Let G be a group. An isomorphism $\phi : G \to G$ is called an **automorphism**. (That is, an automorphism is an isomorphism from a group to itself.)

Fact (that you proved): Let G be a group and $g \in G$. The function $f_g : G \to G$ defined as

$$f_g(x) = gxg^{-1}$$

is an automorphism of G.

Definition 2: Let G be a group. An automorphism $f : G \to G$ defined by $f_g(x) = gxg^{-1}$ is called an **inner automorphism.**

1. (a) Prove that if G is a group with subgroup H and $g \in G$, then the set $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G.

   Proof:

   (b) Prove that if a group G has exactly one subgroup H of order k, than H must be normal in G.

   Proof:

2. (a) Let $G = S_3$ and $g = (12)$. Describe the inner automorphism f_g by filling out the table below. (Note that I filled out one row for you.)

x	$f_g(x)$
$()$	$(12)()(12) = ()$
(12)	
(13)	
(23)	
(123)	
(132)	

(b) Let $G = \mathbb{Z}_3$ and $g = 1$. Describe the inner automorphism f_g by filling out the table below.

x	$f_g(x)$
0	
1	
2	

(c) If G is abelian, what can you conclude about inner automorphisms of G? Justify your answer.

Answer:

(d) Let $G = \mathbb{Z}_3$. Describe an automorphism of G that is not an inner automorphism.

x	$f_g(x)$
0	
1	
2	

(e) You have shown that some automorphisms can be constructed as inner automorphisms, but not all are of that form. Let $Aut(G)$ be the set of all automorphisms of the group G. Prove that this set forms a group under the operation of function composition. (That is, you are proving that $Aut(G) \leq S_G$.)

Proof:

3. Recall that the elements of group $\mathbb{Z}_n$ are equivalence classes. Specifically, $\mathbb{Z}_8 = \{[0], [1], \ldots, [7]\}$ where $[k] = \{k + 8n : n \in \mathbb{Z}\}$ even though we often actually write $\mathbb{Z}_8 = \{0, 1, 2, \ldots, 7\}$. Similarly, $\mathbb{Z}_{20} = \{[0], [1], \ldots, [19]\}$ where $[k] = \{k + 20n : n \in \mathbb{Z}\}$.

   For clarity, you may want to use $[k]_8$ and $[k]_{20}$ to distinguish between whether an equivalence class with representative k is an element in $\mathbb{Z}_8$ or $\mathbb{Z}_{20}$.

   (a) Show that the map $f : \mathbb{Z}_8 \to \mathbb{Z}_{20}$ defined as $f(k) = 4k$ is **not** well-defined. (That is, different representatives of $[k]_8$ may be mapped to different elements of $\mathbb{Z}_{20}$.)

   Answer:

   (b) Show that the map $f : \mathbb{Z}_8 \to \mathbb{Z}_{20}$ defined as $f(k) = 5k$ **is** well-defined.

   Answer:

   (c) Prove that the map $f : \mathbb{Z}_8 \to \mathbb{Z}_{20}$ defined as $f(k) = 5k$ is a group homomorphism and determine the kernel and image of f.

   kernel:

   image:

4. For each map below, determine if it is a homomorphism. (You don't have to **prove** it is or isn't a homomorphism.) If it is a homomorphism, determine its kernel and its image.

   (a) $\phi : \mathbb{R}^* \to GL_2(\mathbb{R})$ defined by $\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$.

   Answer:

   (b) $\phi : \mathbb{R} \to GL_2(\mathbb{R})$ defined by $\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$.

   Answer:

   (c) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}$ defined by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$

   Answer:

   (d) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$

   Answer:

   (e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b$ where $\mathbb{M}_2(\mathbb{R})$ is the additive group of 2×2 matrices with entries in $\mathbb{R}$.

   Answer:

5. Let A be an $m \times n$ matrix. Show that matrix multiplication, $x \to Ax$, defines a homomorphism $\phi : \mathbb{R}^n \mapsto \mathbb{R}^m$.

   Proof:

6. If G is an abelian group and $n \in \mathbb{N}$, show that $\phi : G \to G$ defined by $g \mapsto g^n$ is a group homomorphism.

   Proof:

7. Show that a homomorphism defined on a cyclic group is completely determine by its action on the generator of the group.

   Proof:

8. If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \times G/K$.

   Proof: