NAME:

Instructions: Answer questions in the space provided. You will be graded both on correctness and presentation. This test has 7 questions worth a total of 100 points.

1. (3 points each) Give examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   (a) A subring S of a ring R such that S is not an ideal of R

   (b) A group G, subgroup H of G and an element $a \in G$ such that $aH \neq Ha$.

   (c) A ring R in which the group of units of R is a proper subset of the non-zero elements of R

   (d) An infinite ring with zero divisors. (State the ring R and an example of a zero divisor.)

   (e) A ring R with a nontrivial ideal A such that A is prime but not maximal

2. (10 points) List all abelian groups of order $225 = 9 \cdot 25$ up to isomorphism. Do not write any isomorphism class more than once. For each distinct group, determine the number of elements of order 3. (Note, a bald answer is acceptable here.)

3. (10 points) Let a be an element in the ring R. Let $S = \{r \in R \mid ar = 0\}$. Is S a subring of R? Prove your answer is correct.

4. (15 points)

   (a) State Lagrange's Theorem

   (b) Use Lagrange's Theorem to prove that the order of each element of a finite group must divide the order of the group.

   (c) Prove that every group of order 63 must have an element of order 3.

5. (20 points)

(a) Recall that D_6 is the group of symmetries of a regular hexagon and the center of D_6 is $Z(D_6) = \{R_0,\ R_{180}\}$. What is the order of the element $R_{60}\,Z(D_6)$ in the factor group $D_6/Z(D_6)$? Show that your answer is correct.

(b) Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ be a group with the usual operation of addition modulo 4 in each coordinate and let $K = \langle(1, 2)\rangle$ be the subgroup of G generated by element $(1, 2)$.

i. List the elements of K.

ii. List the elements of G/K.

iii. Is G/K isomorphic to any of the following groups?

D_4(the symmetries of a square), $\mathbb{Z}_6$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2$

Explain your answer.

6. (15 points) Define the mapping $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ as $\phi(a, b) = a - b$.

   (a) Prove that ϕ is a group homomorphism.

   (b) Find the kernel of ϕ.

   (c) Find $\phi^{-1}(3)$.

7. (15 points)

   (a) State the definition of a field F.

   (b) Prove that if F is a nontrivial field, then F has exactly two ideals.

   (c) Prove that if R is a commutative ring with unity such that the only ideals of R are $\{0\}$ and R, then R must be a field.