NAME:

Instructions: Answer questions in the space provided. You will be graded both on correctness and presentation. Note that some problems may ask you to prove theorems stated in your text. Such questions require you to construct a proof, not reference the statement of the Theorem. This test has 6 questions worth a total of 100 points.

1. (3 points each) Give examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   (a) a nonabelian group G and nontrivial subgroup H of G such that $H \triangleleft G$

   (b) a nonabelian group of order 7

   (c) a group G whose only subgroups are G and $\{e\}$

   (d) three nonisomorphic groups of order 44

   (e) a subring S of a ring R such that S is *not* an ideal of R

   (f) a *nontrivial* ring homomorphism from the ring $2\mathbb{Z}$ to $3\mathbb{Z}$

   (g) a maximal ideal in $\mathbb{Q}[x]$

   (h) an infinite ring R such that $\text{char}(R) = 15$

2. (15 points total) Fill in the following FOUR blanks with the correct statement.

   (a) Suppose $A = \langle x \rangle$ is a finite cyclic group of finite order n such that $d \mid n$ (where d is a positive integer.
   Then A has exactly __________ subgroup(s) of order d and exactly __________ element(s) of order d.

   (b) Let $\sigma = (123)(234)(345) \in S_5$.

      i. The order of σ is __________.

      ii. The element σ (is/is not) __________ an element in A_5.

   (c) The order of the element $8 + \langle 6 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 6 \rangle$ is __________.

3. (16 points) Note that here $\mathbb{Z}_n$ will denote a group under the operation of addition modulo n.

   (a) Determine all group homomorphisms from $\mathbb{Z}_{20}$ **onto** $\mathbb{Z}_{10}$. Explain your answer. (Note: the "onto" really does mean *surjective*.)

   (b) Determine all group homomorphisms from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{14}$. Explain your answer. (Note: these homomorphism are not required to be one-to-one or onto.)

4. (15 points)

   (a) State the definition of an ideal I in ring R.

   (b) Prove that if A and B are ideals in the ring R, then the *sum* of A and B,

$$A + B = \{a + b \mid a \in A,\ b \in B\},$$

   is also an ideal in R.

5. (15 points)

   (a) Define *prime ideal.*

   (b) Prove that if R is an commutative ring with unity and A is a prime ideal of R, then R/A is an integral domain.

6. (15 points) Let $\phi : R \to S$ be a ring homomorphism such that the image of ϕ is not $\{0_S\}$. (Another way to say it is that ϕ is NOT the trivial ring homomorphism that sends all elements to 0_S.)

(a) If R has unity and S is an integral domain, show that ϕ carries the unity of R to the unity of S.

(b) Give an example to show that the statement in part (a) need not be true if S is not an integral domain.