NAME:

Instructions: Answer questions in the space provided. You will be graded both on correctness and presentation. Note that some problems may ask you to prove theorems stated in your text. Such questions require you to construct a proof, not reference the statement of the Theorem. This test has 7 questions worth a total of 100 points.

1. (10 points) List up to isomorphism all abelian groups of order $392 = 2^3 \cdot 7^2$. Do not list any group more than once. An answer without explanation is sufficient here.

2. (3 points each) Give examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   (a) cyclic groups G_1 and G_2 such that $G_1 \oplus G_2$ is not cyclic

   (b) an infinite nonabelian group

   (c) a nontrivial normal subgroup of a nonabelian group

   (d) an infinite group such that every element of the group has finite order

   (e) a ring that is not an integral domain

   (f) an integral domain that is not a field

   (g) a finite field

   (h) a prime ideal I in ring R that is not maximal

3. (16 points)

   (a) Write the permutation $\alpha = (14256)(24)(36512)$ as a product of disjoint cycles.

   (b) Determine whether α is even or odd.

   (c) Determine $|\alpha|$.

   (d) Let $H = \{\alpha \in S_n \mid \alpha(1) = 1\}$. (That is, H consists of the subset of permutations of S_n that fix the element 1. For example the permutation $(234)(12)(21)$ fixes 1 but the permutation (123) does not.)
   Prove that H is a subgroup of S_n.

4. (10 points) Let G be a group such that $|G : Z(G)| = 4$. Prove that $G/Z(G) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

5. (16 points) Let R be a ring with a unity, 1, and let $\phi : R \to S$ be a ring homomorphism from R onto the nontrivial ring S.

   (a) Prove that $\phi(1)$ must be the unity of S.

   (b) Show that part (a) above does not hold if ϕ is not onto.

6. (12 points) In both parts of the problem below, $\mathbb{Z}_n$ is a ring.

   (a) Let $a \in \mathbb{Z}_n$, a ring, such that $a^2 = a$. Prove that the function $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ defined as $\phi(z) = az$ is a ring homomorphism.

   (b) Prove that every ring homomorphism ϕ from $\mathbb{Z}_n$ to $\mathbb{Z}_n$ has the form $\phi(z) = az$ where $a^2 = a$.

7. (12 points)

   (a) Prove that if F is a nontrivial field, then F has exactly two ideals.

   (b) Prove that if R is a commutative ring with unity such that the only ideals in R are $\{0\}$ and R, then R is a field.