NAME:

Instructions: Answer questions in the space provided. You will be graded both on correctness and presentation. This test has 6 questions worth a total of 100 points.

1. (3 points each) Give examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   (a) A group G of order at least 3 such that the only subgroups of G are e and G.

   (b) A group G with two elements $a, b \in G$ such that $|a| < \infty$ and $|b| = \infty$

   (c) A non-cyclic Abelian group.

   (d) An element of order 15 in S_8.

   (e) An infinite, noncyclic group.

   (f) Two nonisomorphic groups of order 18.

2. (16 points) Consider the permutation group S_8, and let $\sigma = (13256)(23)(78)(46512)$.

   (a) Express σ as a product of disjoint cycles.

   (b) Express σ as a product of transpositions.

   (c) Give, in disjoint cycle notation, the element σ^{101}.

3. (16 points) Consider the cyclic group G of order 24 generated by a. (So $G = \langle a \rangle$.)

   (a) State a necessary and sufficient condition for an element a^k to generate G.

   (b) State explicitly all generators of G.

   (c) Use Fundamental Theorem of Cyclic Groups to give the orders of all subgroups of G.

4. (20 points)

   (a) State the definition of a *group.*

   (b) Let G be an Abelian group and let H be the subset of elements of finite order from G. Prove that $H \leq G$.

5. (20 points)

(a) State the definition of a *group isomorphism*.

(b) Let G be a group. Show that $\phi : G \to G$ defined by $\phi(g) = g^{-1}$ is an isomorphism if and only if G is Abelian.

6. (10 points) Prove that every group of order 4 is Abelian.