NAME:

Instructions: Answer questions in the space provided. You will be graded both on correctness and presentation. This test has 7 questions worth a total of 100 points.

1. (3 points each) Give examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   (a) an infinite non-Abelian group G and a proper, nontrivial, normal subgroup $N \triangleleft G$.

   (b) a group G and nontrivial subgroup $H \leq G$ so that $|G : H| = 3$.

   (c) a homomorphism $\phi : G \to G'$ of groups that is not an isomorphism. (Indicate G, G' and ϕ explicitly.)

   (d) a group G, a subgroup $H \leq G$, and an element $a \in G$ so that $aH \neq Ha$ (i.e.,the right and left cosets of H in G are unequal.)

   (e) a non-trivial group homomorphism $\phi : Z_{12} \to Z_5$.

2. (5 points) List all cosets of $\langle 4 \rangle$ in $\mathbb{Z}$.

3. (10 points) Use the Fundamental Theorem of Finite Abelian Groups to list, up to isomorphism, all Abelian groups of order $756 = 2^2 \cdot 3^3 \cdot 7$. You do not need to justify your answer here; simply give a complete list without repetitions.

4. (15 points)

   (a) Give the definition of a group homomorphism $\phi : G \to H$.

   (b) Let $\phi : G \to H$ be a group homomorphism of finite groups that is onto. Prove that if H has an element of order 8, then G has an element of order 8.

5. (20 points) Let $\mathbb{Z}$ and $\mathbb{Q}$ be the usual groups under the operation of addition.

   (a) Explain why it is immediate that $\mathbb{Z} \lhd \mathbb{Q}$.

   (b) Describe briefly the elements in the factor group $\mathbb{Q}/\mathbb{Z}$ under addition and give a specific, nontrivial example of an element in $\mathbb{Q}/\mathbb{Z}$.

   (c) Prove that $\mathbb{Q}/\mathbb{Z}$ is infinite.

   (d) Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

6. (15 points)

   (a) State Lagrange's Theorem.

   (b) Use Lagrange's Theorem to prove that all groups of order p, where p is a prime, are cyclic.

7. (10 points) Let G be a finite group and let p be a prime. If $p^2 > |G|$, prove that any subgroup of order p is normal in G.