NAME: Solutions

1. (3 points each) Give examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   NOTE: There are many possible examples. I chose the ones I thought were the easiest or most obvious.

   (a) an infinite non-Abelian group G and a proper, nontrivial, normal subgroup $N \triangleleft G$.

   $SL(2\mathbb{R}) \triangleleft GL(2, \mathbb{R})$

   (b) a group G and nontrivial subgroup $H \leq G$ so that $|G : H| = 3$.

   $G = \mathbb{Z}_6,\ H = \langle 3 \rangle$

   (c) a homomorphism $\phi : G \to G'$ of groups that is not an isomorphism. (Indicate G, G' and ϕ explicitly.)

   $\phi : \mathbb{Z}_3 \to \mathbb{Z}_3$ defined by $\phi(x) = 0$.

   (d) a group G, a subgroup $H \leq G$, and an element $a \in G$ so that $aH \neq Ha$ (i.e.,the right and left cosets of H in G are unequal.)

   $G = S_3$, $H = \{(1), (12)\}$, and $a = (123)$. Then $(123)H = \{(123), (13)\}$ and $H(123) = \{(123), (23)\}$.

   (e) a non-trivial group homomorphism $\phi : Z_{12} \to Z_5$.

   Only the trivial homomorphism is possible. For any homomorphism $\phi : Z_{12} \to Z_5$, $|\phi(\mathbb{Z}_{12})|$ must divide $|\mathbb{Z}_5| = 5$ and $|\mathbb{Z}_{12}| = 12$. So $|\phi(\mathbb{Z}_{12})| = \gcd(5, 12) = 1$.

2. (5 points) List all cosets of $\langle 5 \rangle$ in $\mathbb{Z}$.

   $\langle 5 \rangle$, $1 + \langle 5 \rangle$, $2 + \langle 5 \rangle$, $3 + \langle 5 \rangle$, $4 = \langle 5 \rangle$

3. (10 points) Use the Fundamental Theorem of Finite Abelian Groups to list, up to isomorphism, all Abelian groups of order $756 = 2^2 \cdot 3^3 \cdot 7$. You do not need to justify your answer here; simply give a complete list without repetitions.
$\mathbb{Z}_4 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_7$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_7$
$\mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$
$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$

4. (15 points)

   (a) Give the definition of a group homomorphism $\phi : G \to H$.
   A mapping, ϕ, from group G to group H is a group homomorphism if ϕ is operation preserving. That is, if for every $a, b \in G$, $\phi(ab) = \phi(a)\phi(b)$.

   (b) Let $\phi : G \to H$ be a group homomorphism of finite groups that is onto. Prove that if H has an element of order 8, then G has an element of order 8.

   [Note that the proof below can be shortened by referencing familiar theorems and corollaries.]

   Let $\phi : G \to H$ be a group homomorphism of finite groups that is onto and let $h \in H$ of order 8. Since ϕ is onto, we know there exists an element $g \in G$ such that $\phi(g) = h$. Since G is finite, $|g|$ must be finite, say $|g| = n$.
   Because ϕ is a homomorphism, $e = \phi(e) = \phi(g^n) = (\phi(g))^n = h^n$. Thus, $8 \mid n$.
   Now we claim that since $n = 8k$, the element g^k has order 8. This follows because we see that if the order of g^k were some integer $m < 8$, $(g^k)^m = e$, contradicting the order of g.

   Why is finiteness necessary here? A counterexample is the efficient and unequivocal way to go.

   Let $\phi : \mathbb{Z} \to \mathbb{Z}_8$.

5. (20 points) Let $\mathbb{Z}$ and $\mathbb{Q}$ be the usual groups under the operation of addition.

   (a) Explain why it is immediate that $\mathbb{Z} \triangleleft \mathbb{Q}$.
   $\mathbb{Q}$ is Abelian so all subgroups are normal subgroups.

   (b) Describe briefly the elements in the factor group $\mathbb{Q}/\mathbb{Z}$ under addition and give a specific, nontrivial example of an element in $\mathbb{Q}/\mathbb{Z}$.
   Elements of the factor group are cosets of the subgroup $\mathbb{Z}$.
   One example: $\frac{1}{2} + \mathbb{Z} = \{\cdots - 1/2,\ 1/2,\ 3/2,\ 5/2, \cdots\}$

(c) Prove that $\mathbb{Q}/\mathbb{Z}$ is infinite.
It is sufficient to describe an infinite family of cosets. In particular, I do not need to *characterize* all cosets, though some people did.

I claim the family $\{\frac{1}{2}+\mathbb{Z},\ \frac{1}{4}+\mathbb{Z},\ \frac{1}{8}+\mathbb{Z},\ \frac{1}{16}+\mathbb{Z},\ \cdots\}$ is an infinite set of *distinct* cosets of $\mathbb{Q}/\mathbb{Z}$.

Assume $\frac{1}{2^{k_1}}+\mathbb{Z}=\frac{1}{2^{k_2}}+\mathbb{Z}$ where $k_1 \le k_2$. Then there exist $n, m \in \mathbb{Z}$ such that $\frac{1}{2^{k_1}}+n=\frac{1}{2^{k_2}}+m$ or, equivalently, $n-m=\frac{1}{2^{k_2}}-\frac{1}{2^{k_1}}$. Since $1 > \frac{1}{2^{k_2}}-\frac{1}{2^{k_1}} \ge 0$ and $n-m \in \mathbb{Z}$, $k_1 = k_2$. So indeed all the cosets in my infinite list are distinct.

(d) Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

Let $r \in \mathbb{Q}$, so $r=\frac{a}{b}$. I claim $|\frac{a}{b}+\mathbb{Z}| \le b$.

Observe that $b\cdot(\frac{a}{b}+\mathbb{Z})=(b\cdot\frac{a}{b})+\mathbb{Z}=a+\mathbb{Z}=\mathbb{Z}$, the identity in $\mathbb{Q}/\mathbb{Z}$.

6. (a) State Lagrange's Theorem.

Let H be a subgroup of the finite group G. Then $|H| \mid |G|$. Further, $|G:H|=|G|/|H|$.

(b) Use Lagrange's Theorem to prove that all groups of order p, where p is a prime, are cyclic.

Let G be a group of order p and let $a \in G-e$. We know $\langle a\rangle \le G$. Thus, by Lagrange's Theorem, $|\langle a\rangle| \mid |G|$. Thus, $|\langle a\rangle| \mid p$. Since $a \ne e$, $|\langle a\rangle| \ge 2$. Since p is prime, $|\langle a\rangle| = p$. But this implies $\langle a\rangle = G$ forcing G to be cyclic.

7. (10 points) Let G be a finite group and let p be a prime. If $p^2 > |G|$, prove that any subgroup of order p is normal in G.

Let G be a finite group with order less than p^2, where p is a prime. Let $H \le G$ of order p. We claim that H is unique.
If there exists $K \le G$ of order p and $K \ne H$, then $K \cap H = e$. Thus, applying the HK Theorem, we get the contradiction:

$$p^2 > |G| \ge |HK| = |H|\cdot|K|/|H\cap K| = |H|\cdot|K| = p^2.$$

Because H is unique, we know that for every $x \in G$, $xHx^{-1}=H$ because ϕ_x defined as $\phi_x(g)=xgx^{-1}$ is an isomorphism and as such will map a subgroup of order p to a subgroup of order p.

Since $xHx^{-1}=H$ for every $x \in G$, we have shown that $H \triangleleft G$.