Name: ____________________

Rules:

You have 2 hours to complete this Final Exam.

Partial credit will be awarded, but you must show your work.

No notes, books, or other aids are allowed.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	20	
3	16	
4	12	
5	12	
6	24	
Extra Credit	5	
Total	100	

1. Let G and H be groups and let $\phi : G \to H$ be a group homomorphism.

   (a) (2 pts) State the definition of a **group homomorphism**.

   (b) (2 pts) State the definition of the **kernel of** ϕ, ker ϕ.

   (c) (8 pts) Prove ker ϕ is a normal subgroup of G. (Note that you must show ker ϕ is a subgroup of G **and** that it is normal.)

2. (18 points) Give an examples of the following, if they exist. Otherwise, briefly explain why such examples do not exist.

   (a) An infinite nonabelian group.

   (b) A nonabelian group of order $n = 11$.

   (c) An infinite group G with multiple elements of finite order.

   (d) Three nonisomorphic groups of order 12.

   (e) A commutative ring with unity that is not an integral domain.

   (f) A ring R and an ideal I such that I is prime.

   (g) A ring R and an ideal I that is maximal in R.

   (h) A ring R such that $R[x]$ contains a unit of degree at least 1.

3. (12 points) Let G be an abelian group. Let $H = \{a \in G : |a| < \infty\}$. (That is, H consists of all elements of G of finite order.)

   Prove that H is a subgroup of G.

4. (a) (4 points) State Lagrange's Theorem

   (b) (10 points) Let G be a group of order pq where p and q are both primes. Prove that every proper subgroup of G is cyclic.

5. Let R be the ring of functions $f : \mathbb{R} \to \mathbb{R}$ with the usual operations of addition and multiplication. Let S be the set of differentiable functions in R. (Note: All the functions in R, and therefore S, have domain $\mathbb{R}$.)

   (a) (10 points) Prove that S is a subring of R.

   (b) (4 points) Prove that S is **not** an ideal of R.

6. Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in \mathbb{Z} \right\}$. Consider the function $\phi : R \to \mathbb{Z}$ defined by $\phi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = a$.

   (a) (10 points) Prove that ϕ is a ring homomorphism.

   (b) (4 points) Determine the kernel of ϕ.

7. (4 points each) Short Answer

   (a) What is the order of the factor group $\mathbb{Z}_{60}/\langle 15\rangle$?

   (b) What is the order of the element $10+\langle 15\rangle$ in the factor group $\mathbb{Z}_{60}/\langle 15\rangle$?

   (c) Is $2x^4+1$ an element of $\langle x^2+2\rangle$, the ideal generated by x^2+2 in $\mathbb{Z}_3[x]$? Justify your answer.

   (d) Show that the map $f(x)=5x$ is **not** a ring homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{60}$.

5 pts Extra Credit: Suppose $f(x)$ is irreducible in $F[x]$, where F is a field. Prove that for every nonzero polynomial $g(x)\in F[x]$, either $\gcd(f(x),g(x))=1$ or $f(x)\,\big|\,g(x)$.