**Name:** ____________________

## Rules:

You have 1.5 hours to complete this midterm.

Partial credit will be awarded, but you must show your work.

No notes, books, or other aids are allowed.

Turn off anything that might go beep during the exam.

Good luck!

| Problem | Possible | Score |
| --- | --- | --- |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 25 | |
| 4 | 25 | |
| 5 | 15 | |
| 6 | 15 | |
| Extra Credit | 5 | |
| Total | 100 | |

1. (10 points) Use the method of induction to prove the statement below.

For all integers $n \geq 1$,

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$$

2. (10 points) Suppose $a, b$, and $c$ are nonzero integers. Prove that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$.

3. (25 points) Give an examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   (a) An infinite nonabelian group.

   (b) An abelian group of order $n$ for any integer $n$, $n \geq 1$.

   (c) A group $G$ with exactly two subgroups.

   (d) An infinite cyclic group.

   (e) A nonabeliean group such that every proper subgroup is abelian.

4. (25 points) Let $\sigma = (12345)$, $\tau = (2436)$, and $\rho = (123)(24)(264)(45)$ be elements of $S_6$, the symmetric group on 6 letters.

   (a) Find $\sigma \circ \tau(2)$ and $\tau \circ \sigma(2)$.

   (b) Determine $|\sigma|$, the order of $\sigma$ in $S_6$.

   (c) Write $\rho$ as a product of disjoint cycles.

   (d) Write $\rho$ as a product of transpositions.

   (e) Write $(\sigma \circ \tau)^{-1}$ as a product of disjoint cycles.

5. (15 points)

   (a) State the definition of a group.

   (b) Let $X$ be the set of bijections from $\mathbb{R}$ to $\mathbb{R}$. Show that the set $X$ under the operation of function composition is a group.

6. (15 points) Short Answer

(a) Determine the order of the element 3 in the group $(\mathbb{Z}_{18}, +)$, the integers under addition modulo 18 and find the inverse of 3.

(b) Identify the elements of $U(9)$, determine the order of $4$, and identify the inverse of $4$.

(c) Let $a$ be an element of $G$, a group. If $a^{12} = e$, what are the possible orders of $a$?

7. (5 points) Prove that if $G$ is a group such that for every $x, y \in G$, $xy = x^{-1}y^{-1}$, then $G$ is abelian.