1. (10 points) (HW 4 Problem 4)Let H and K be subgroups of the group G. Recall that $HK = \{hk : h \in H \text{ and } k \in K\}$. Prove that if G is abelian, then $HK \leq G$.

Answer: Let H and K be subgroups of the abelian group G. Let $HK = \{hk : h \in H \text{ and } k \in K\}$.
(Show HK is closed.) Let $h_1k_1, h_2k_2 \in HK$. Since G is abelian, $h_1k_1h_2k_2 = h_1h_2k_1k_2 = h_3k_3$, where $h_3 \in H$ and $k_3 \in K$. Thus, $h_1k_1h_2k_2 \in HK$.
(Show $e \in HK$.) Since H and K are subgroups, $e \in H$ and $e \in K$. Thus, $e = ee \in HK$.
(Show HK contains inverses.) Let $hk \in HK$ where $h \in H$ and $k \in K$. Since H and K are subgroups, $h^{-1} \in H$ and $k^{-1} \in K$. Thus, $h^{-1}k^{-1} = HK$. Since G is abelian and using our knowledge of inverses, we know that $h^{-1}k^{-1} = k^{-1}h^{-1} = (hk)^{-1}$. Thus, $(hk)^{-1} \in S$.
Since we have shown that HK is closed, contains the identity and inverses, HK is a subgroup.

2. (16 points) Give an examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

(a) a noncyclic group of order 13

None exists since all groups of prime order are cyclic.

(b) an infinite group G and element $g \in G$, such that $|g|$ is finite
In retrospect, I should have added the word **nontrivial** element g.

$G = \mathbb{Z}_2 \times \mathbb{Z}$ and element $(1,0)$.

(c) two nonisomorphic groups of order 18

many examples here: D_9 or $\mathbb{Z}_{18}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$

(d) a nonabelian group G and a subgroup H of G such that $H \lhd G$

many examples here but the easiest is $G = S_n$ and $H = A_n$

3. (20 points)

   (a) Suppose that H is a subgroup of the group G. Define the **index** of H in G.

   The index of H in G is the number of left (or right) cosets of H.

   (b) Give an example of a group G and a **nontrivial** subgroup H such that the index of H in G is 50.

   Let $G = \mathbb{Z}_{100}$ and $H = \{1, 50\}$.

   (c) (HW 5 Problem 7) Let H be a subgroup of the group G. Prove that if $[G : H] = 2$, then for every $a, b \in G \backslash H$, $ab \in H$.

   Proof: Let H be a subgroup of the group G of index 2. Let $a, b \in G \backslash H$. Since $a \notin H$, it follows that $a^{-1} \notin H$. Since $a^{-1}, b \notin H$, we know $a^{-1}H \neq H$ and $bH \neq H$. Since $[G : H] = 2$, $a^{-1}H = bH$. By Lemma 6.3, we know that if $a^{-1}H = bH$, then $(a^{-1})^{-1}b = ab \in H$.

4. (15 points) ((HW 6 Problem 9) Let G be a group and $g \in G$. Define the function $f_g(x) : G \to G$ by $f_g(x) = gxg^{-1}$. Prove that f_g is an isomorphism from G to itself.

Proof:
(Show f_g is 1-1.) Let $x, y \in G$ such that $f_g(x) = f_g(y)$. Then by the definition of f_g, $gxg^{-1} = gyg^{-1}$. Using left and right cancellation, we obtain $x = y$.

(Show f_g is onto.) Let $y \in G$. Pick $x = g^{-1}yg \in G$. Then $f_g(x) = f_g(g^{-1}yg) = gg^{-1}ygg^{-1} = y$.

(Show f_g respects op.) Let $x, y \in G$. Observe

$$\begin{aligned} f(xy) &= gxyg^{-1} && \text{by the definition of } G \\ &= gxeyg^{-1} && \text{by the definition of } e \\ &= gxg^{-1}gyg^{-1} && \text{since } e = g^{-1}g \\ &= f(x)f(y) && \text{by the definition of } f. \end{aligned}$$

5. (12 points)

(a) State Lagrange's Theorem

Suppose that H is a subgroup of the group G. Then,

- $[G:H] = \dfrac{|G|}{|H|}$ and
- $|H| \, \big| \, |G|$

(b) Suppose K is a proper subgroup of H and H is a proper subgroup of G. (So $K < H < G$.) If $|K| = 10$ and $|G| = 200$, what are the possible orders of H? (Or, what are the possible values for $|H|$?) Explain your reasoning.

Since $K \leq H$ and $|K| = 10$, Lagrange's Theorem implies that $|H| = 10n$ for some integer n. Since K is a **proper** subgroup, $n \geq 2$. Since H is a subgroup of G and $|G| = 200$, Lagrange's Theorem implies that $|H| \, \big| \, |G|$ or, equivalently, $10n \big| 200$. Since H is a **proper** subgroup of G, we conclude that $n < 20$.

Using the fact that $200 = 2^3 5^2 = 10 \cdot 2^2 \cdot 5$, we conclude the possible values of n are: 2,4,5, and 10. So, the possible values of $|H|$ are 20, 40, 50, and 100.

6. (12 points)

(a) State the definition of a **normal subgroup.**

The subgroup H of G is called **normal** if for every $g \in G$, $gH = Hg$.

(b) Let $G = GL_2(\mathbb{R})$ and let $H = \{A \in GL_2(\mathbb{R}) : \det(A) = 2^k \text{ for } k \in \mathbb{Z}\}$ be a subgroup of G. Prove that $H \lhd G$.
Note: You do not need to prove that H is a subgroup of G. You only need to prove that H is normal in G.

Proof: Let $G = GL_2(\mathbb{R})$ and let $H = \{A \in GL_2(\mathbb{R}) : \det(A) = 2^k \text{ for } k \in \mathbb{Z}\}$. By Theorem 10.3, we know that $H \lhd G$ if and only if $ghg^{-1} \in H$ for every $g \in G$ and every $h \in H$.
Let $B \in GL_2(\mathbb{R})$ and $A \in H$. We need to show that $BAB^{-1} \in H$. Using properties of the determinant, we know that

$$det(BAB^{-1}) = det(B)det(A)det(B^{-1}) = det(B)det(B^{-1})det(A) = 1 \cdot det(A).$$

But $det(A) = 2^k$ for some integer k, since $A \in H$. Thus, $det(BAB^{-1}) = 2^k$ for some integer k. Thus, $BAB^{-1} \in H$. Thus, $H \lhd G$.

7. (15 points) Let $G = \mathbb{Z}_{24}$ and $H = \langle 8 \rangle$.

   (a) In one or two sentences, explain why $H \lhd G$.

   G is abelian so all subgroups of G are normal.

   (b) List the distinct elements of G/H.

   $0+H, 1+H, 2+H, 3+H, 4+H, 5+H, 6+H, 7+H$

   (c) For cosets $10+H$ and $20+H$, determine $(10+H)+(20+H)$

   $30+H = 6+H$

   (d) Determine the order of the element $10+H$ in G/H.

   Since $10+H = 2+H$, we can see that $4 \cdot (2+H) = 0+H$ and 4 is the smallest such value. So $|2+H| = 4$.

   (e) Can G/H have an element of order 5? If so, find such an element. If not, explain why it is not possible.

   Answer: No.
   Reasoning: Since the order of G/H is 8, the order of every element must divide 8.

Extra Credit: (5 points) Suppose that H and K are subgroups of the group G. Prove that if there exist elements $a, b \in G$ such that $aH \subseteq bK$, then $H \subseteq K$.

Proof: Suppose that H and K are subgroups of the group G. Suppose there exist elements $a, b \in G$ such that $aH \subseteq bK$.

Let $h \in H$. We need to show that $h \in K$.

Since $a = ae \in aH$ and $aH \subseteq bK$, we know there exists some $k \in K$ such that $a = bk$ and, consequently, $a^{-1} = k^{-1}b^{-1}$.

Since $h \in H$ and $aH \subseteq bK$, we know there exists some $k' \in K$ such that $ah = bk'$. Operating on the left by a^{-1}, we conclude

$$h = a^{-1}bk' = k^{-1}b^{-1}bk' = k^{-1}k'.$$

Since K is a group, $k^{-1}k'$ is in K. Thus, $h \in K$.