Name: ____________________

Rules:

You have 1.5 hours to complete this midterm.

Partial credit will be awarded, but you must show your work.

No notes, books, or other aids are allowed.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	16	
3	20	
4	15	
5	12	
6	12	
7	15	
Extra Credit	5	
Total	100	

1. (10 points) Let H and K be subgroups of the group G. Recall that $HK = \{hk : h \in H \text{ and } k \in K\}$. Prove that if G is abelian, then $HK \leq G$.

2. (16 points) Give an examples of the following, if they exist. Otherwise briefly explain why such examples do not exist.

   (a) a noncyclic group of order 13

   (b) an infinite group G and element $g \in G$, such that $|g|$ is finite

   (c) two nonisomorphic groups of order 18

   (d) a nonabelian group G and a subgroup H of G such that $H \lhd G$

3. (20 points)

   (a) Suppose that H is a subgroup of the group G. Define the **index** of H in G.

   (b) Give an example of a group G and a **nontrivial** subgroup H such that the index of H in G is 50.

   (c) Let H be a subgroup of the group G. Prove that if $[G:H]=2$, then for every $a,b \in G \backslash H$, $ab \in H$.

4. (15 points) Let G be a group and $g \in G$. Define the function $f_g(x) : G \to G$ by $f_g(x) = gxg^{-1}$. Prove that f_g is an isomorphism from G to itself.

5. (12 points)

   (a) State Lagrange's Theorem

   (b) Suppose K is a proper subgroup of H and H is a proper subgroup of G. (So $K < H < G$.) If $|K| = 10$ and $|G| = 200$, what are the possible orders of H? (Or, what are the possible values for $|H|$?) Explain your reasoning.

6. (12 points)

   (a) State the definition of a **normal subgroup.**

   (b) Let $G = GL_2(\mathbb{R})$ and let $H = \{A \in GL_2(\mathbb{R}) : \det(A) = 2^k \text{ for } k \in \mathbb{Z}\}$ be a subgroup of G. Prove that $H \lhd G$.
   Note: You do not need to prove that H is a subgroup of G. You only need to prove that H is normal in G.

7. (15 points) Let $G = \mathbb{Z}_{24}$ and $H = \langle 8 \rangle$.

   (a) In one or two sentences, explain why $H \lhd G$.

   (b) List the distinct elements of G/H.

   (c) For cosets $10 + H$ and $20 + H$, determine $(10 + H) + (20 + H)$.

   (d) Determine the order of the element $10 + H$ in G/H.

   (e) Can G/H have an element of order 5? If so, find such an element. If not, explain why it is not possible.

Extra Credit: (5 points) Suppose that H and K are subgroups of the group G. Prove that if there exist elements $a, b \in G$ such that $aH \subseteq bK$, then $H \subseteq K$.